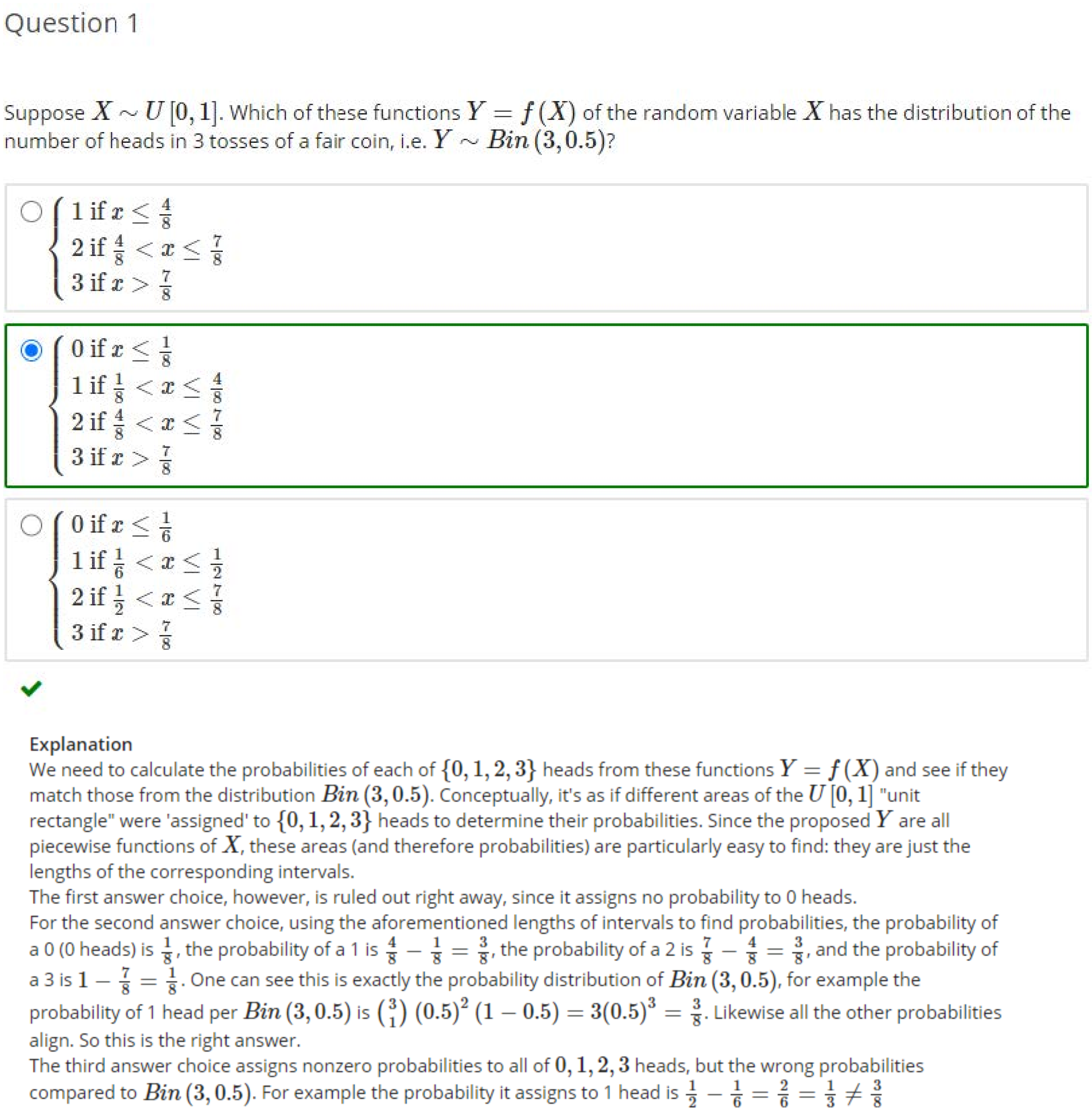
Functions of Random Variables (16 Questions)



Question 1

Suppose $X \sim U[0,1]$. Which of these functions $Y = f(X)$ of the random variable X has the distribution of the number of heads in 3 tosses of a fair coin, i.e. $Y \sim Bin(3, 0.5)$?

○ $\begin{cases} 1 \text{ if } x \leq \frac{4}{8} \\ 2 \text{ if } \frac{4}{8} < x \leq \frac{7}{8} \\ 3 \text{ if } x > \frac{7}{8} \end{cases}$

● $\begin{cases} 0 \text{ if } x \leq \frac{1}{8} \\ 1 \text{ if } \frac{1}{8} < x \leq \frac{4}{8} \\ 2 \text{ if } \frac{4}{8} < x \leq \frac{7}{8} \\ 3 \text{ if } x > \frac{7}{8} \end{cases}$

○ $\begin{cases} 0 \text{ if } x \leq \frac{1}{6} \\ 1 \text{ if } \frac{1}{6} < x \leq \frac{1}{2} \\ 2 \text{ if } \frac{1}{2} < x \leq \frac{7}{8} \\ 3 \text{ if } x > \frac{7}{8} \end{cases}$

✓

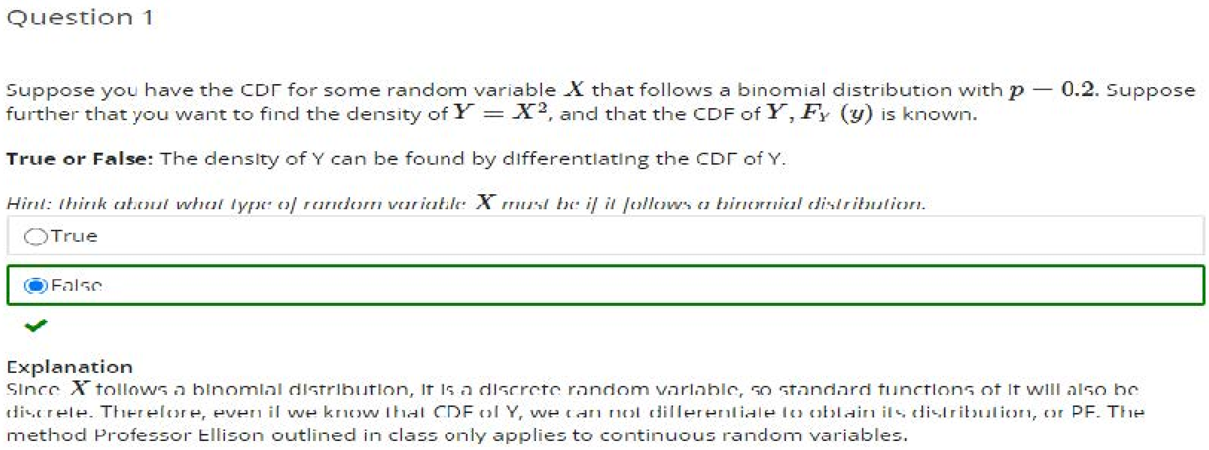
Explanation

We need to calculate the probabilities of each of $\{0,1,2,3\}$ heads from these functions $Y = f(X)$ and see if they match those from the distribution $Bin(3, 0.5)$. Conceptually, it's as if different areas of the $U[0,1]$ "unit rectangle" were 'assigned' to $\{0,1,2,3\}$ heads to determine their probabilities. Since the proposed Y are all piecewise functions of X, these areas (and therefore probabilities) are particularly easy to find: they are just the lengths of the corresponding intervals.

The first answer choice, however, is ruled out right away, since it assigns no probability to 0 heads.

For the second answer choice, using the aforementioned lengths of intervals to find probabilities, the probability of a 0 (0 heads) is $\frac{1}{8}$, the probability of a 1 is $\frac{4}{8} - \frac{1}{8} = \frac{3}{8}$, the probability of a 2 is $\frac{7}{8} - \frac{4}{8} = \frac{3}{8}$, and the probability of a 3 is $1 - \frac{7}{8} = \frac{1}{8}$. One can see this is exactly the probability distribution of $Bin(3, 0.5)$, for example the probability of 1 head per $Bin(3, 0.5)$ is $\binom{3}{1}(0.5)^2(1-0.5) = 3(0.5)^3 = \frac{3}{8}$. Likewise all the other probabilities align. So this is the right answer.

The third answer choice assigns nonzero probabilities to all of $0, 1, 2, 3$ heads, but the wrong probabilities compared to $Bin(3, 0.5)$. For example the probability it assigns to 1 head is $\frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \neq \frac{3}{8}$



Question 1

Suppose you have the CDF for some random variable X that follows a binomial distribution with $p - 0.2$. Suppose further that you want to find the density of $Y = X^2$, and that the CDF of Y, $F_Y(y)$ is known.

True or False: The density of Y can be found by differentiating the CDF of Y.

Hint: think about what type of random variable X must be if it follows a binomial distribution.

○ True

● False

✓

Explanation

Since X follows a binomial distribution, it is a discrete random variable, so standard functions of it will also be discrete. Therefore, even if we know that CDF of Y, we can not differentiate to obtain its distribution, or PF. The method Professor Ellison outlined in class only applies to continuous random variables.

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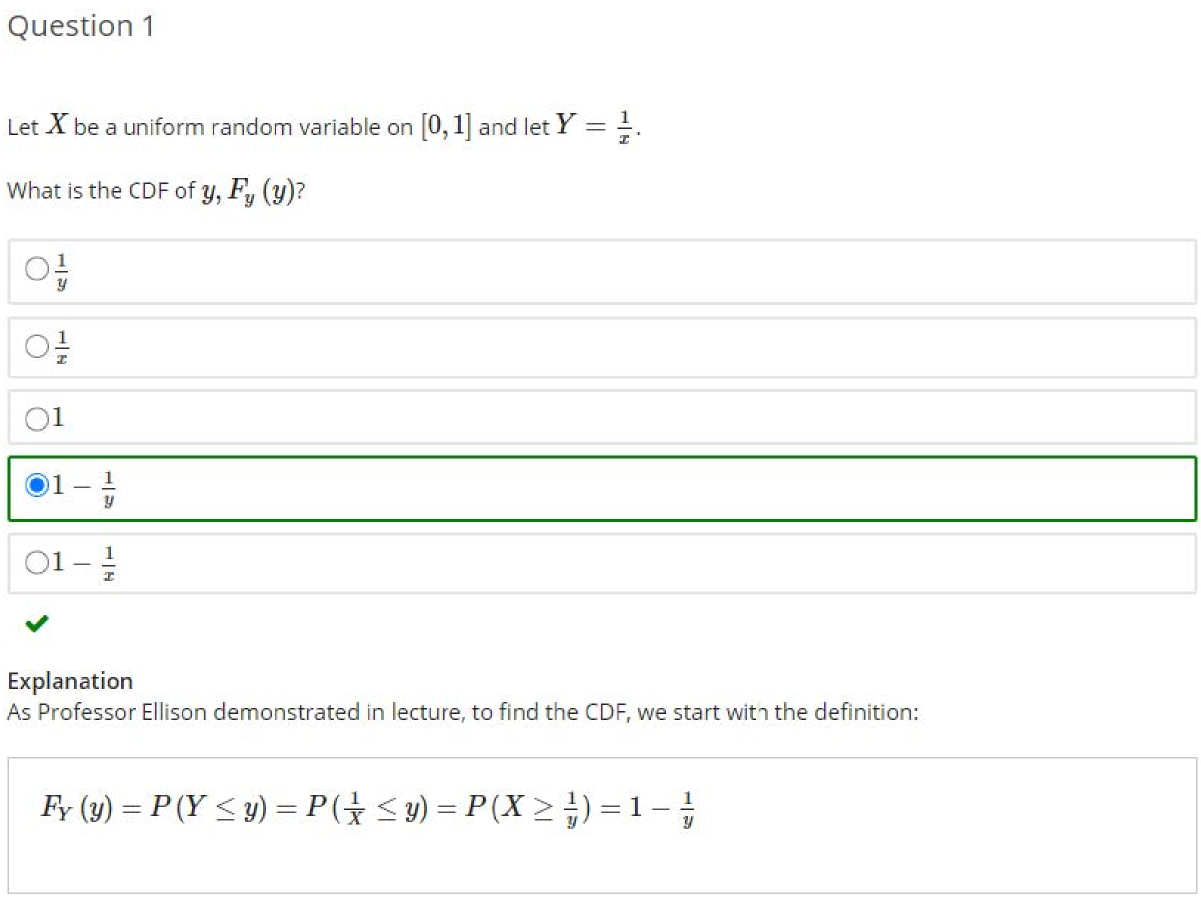


Question 1

Let X be a uniform random variable on $[0,1]$ and let $Y = \frac{1}{x}$.

What is the CDF of y, $F_y\left(y\right)$?

○ $\frac{1}{y}$

○ $\frac{1}{x}$

○ 1

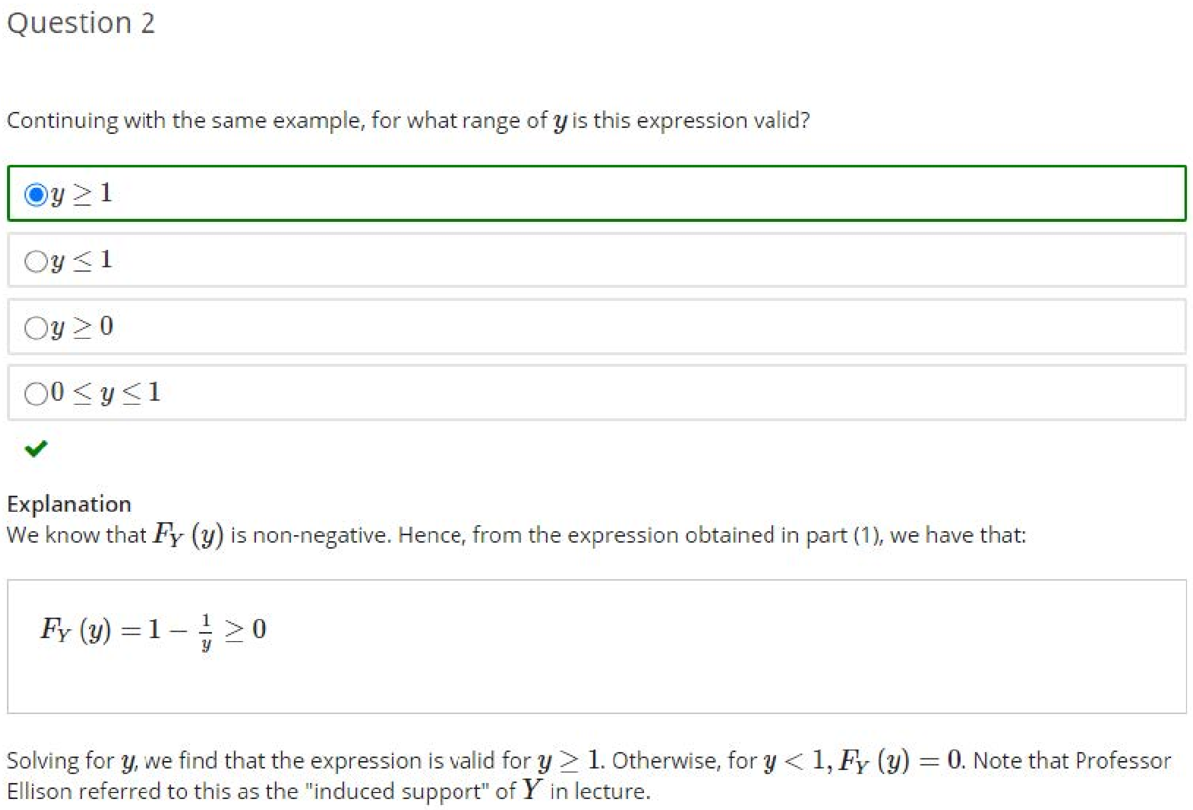
● $1 - \frac{1}{y}$

○ $1 - \frac{1}{x}$

✓

Explanation
As Professor Ellison demonstrated in lecture, to find the CDF, we start with the definition:

$$F_Y\left(y\right) = P\left(Y \leq y\right) = P\left(\frac{1}{X} \leq y\right) = P\left(X \geq \frac{1}{y}\right) = 1 - \frac{1}{y}$$



Question 2

Continuing with the same example, for what range of y is this expression valid?

● $y \geq 1$

○ $y \leq 1$

○ $y \geq 0$

○ $0 \leq y \leq 1$

✓

Explanation
We know that $F_Y\left(y\right)$ is non-negative. Hence, from the expression obtained in part (1), we have that:

$$F_Y\left(y\right) = 1 - \frac{1}{y} \geq 0$$

Solving for y, we find that the expression is valid for $y \geq 1$. Otherwise, for $y < 1, F_Y\left(y\right) = 0$. Note that Professor Ellison referred to this as the "induced support" of Y in lecture.

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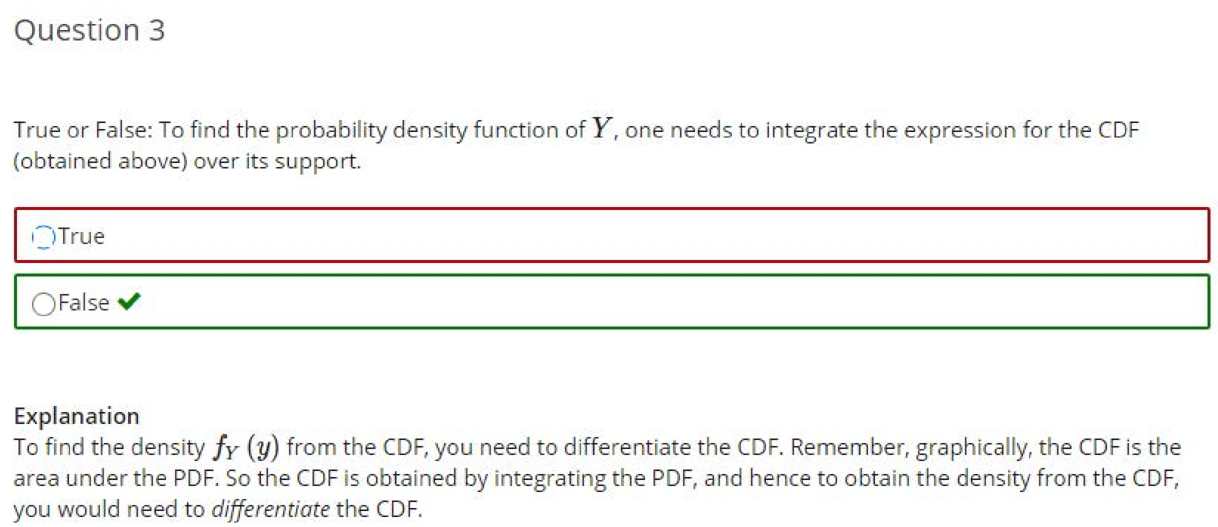


Question 3

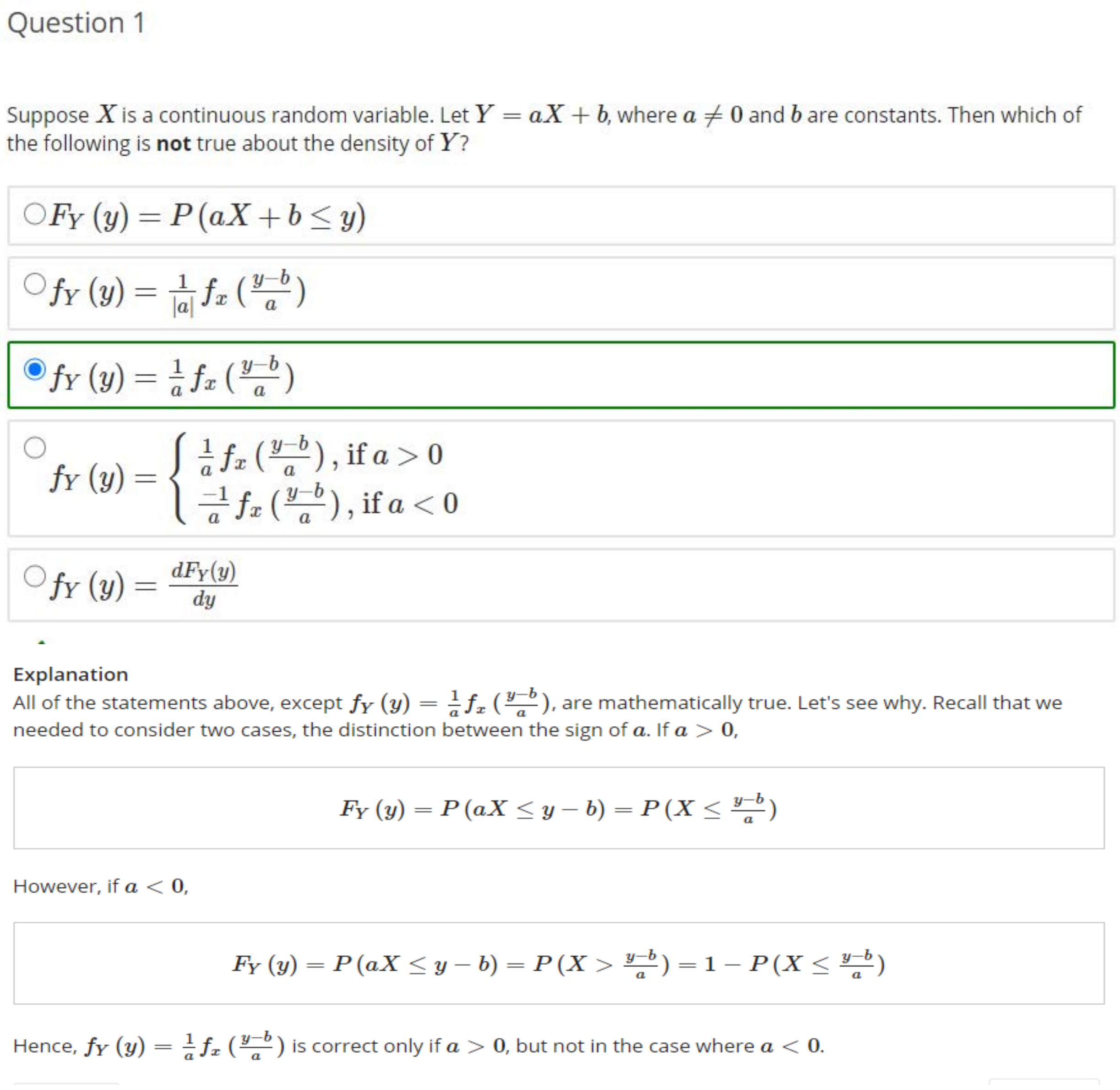
True or False: To find the probability density function of Y, one needs to integrate the expression for the CDF (obtained above) over its support.

○ True

○ False ✓

Explanation

To find the density $f_Y(y)$ from the CDF, you need to differentiate the CDF. Remember, graphically, the CDF is the area under the PDF. So the CDF is obtained by integrating the PDF, and hence to obtain the density from the CDF, you would need to *differentiate* the CDF.



Question 1

Suppose X is a continuous random variable. Let $Y = aX + b$, where $a \neq 0$ and b are constants. Then which of the following is **not** true about the density of Y?

○ $F_Y(y) = P(aX + b \leq y)$

○ $f_Y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right)$

● $f_Y(y) = \frac{1}{a} f_x\left(\frac{y-b}{a}\right)$

○ $f_Y(y) = \begin{cases} \frac{1}{a} f_x\left(\frac{y-b}{a}\right), \text{if } a > 0 \\ \frac{-1}{a} f_x\left(\frac{y-b}{a}\right), \text{if } a < 0 \end{cases}$

○ $f_Y(y) = \frac{dF_Y(y)}{dy}$

Explanation

All of the statements above, except $f_Y(y) = \frac{1}{a} f_x\left(\frac{y-b}{a}\right)$, are mathematically true. Let's see why. Recall that we needed to consider two cases, the distinction between the sign of a. If $a > 0$,

$$F_Y(y) = P(aX \leq y - b) = P\left(X \leq \frac{y-b}{a}\right)$$

However, if $a < 0$,

$$F_Y(y) = P(aX \leq y - b) = P\left(X > \frac{y-b}{a}\right) = 1 - P\left(X \leq \frac{y-b}{a}\right)$$

Hence, $f_Y(y) = \frac{1}{a} f_x\left(\frac{y-b}{a}\right)$ is correct only if $a > 0$, but not in the case where $a < 0$.

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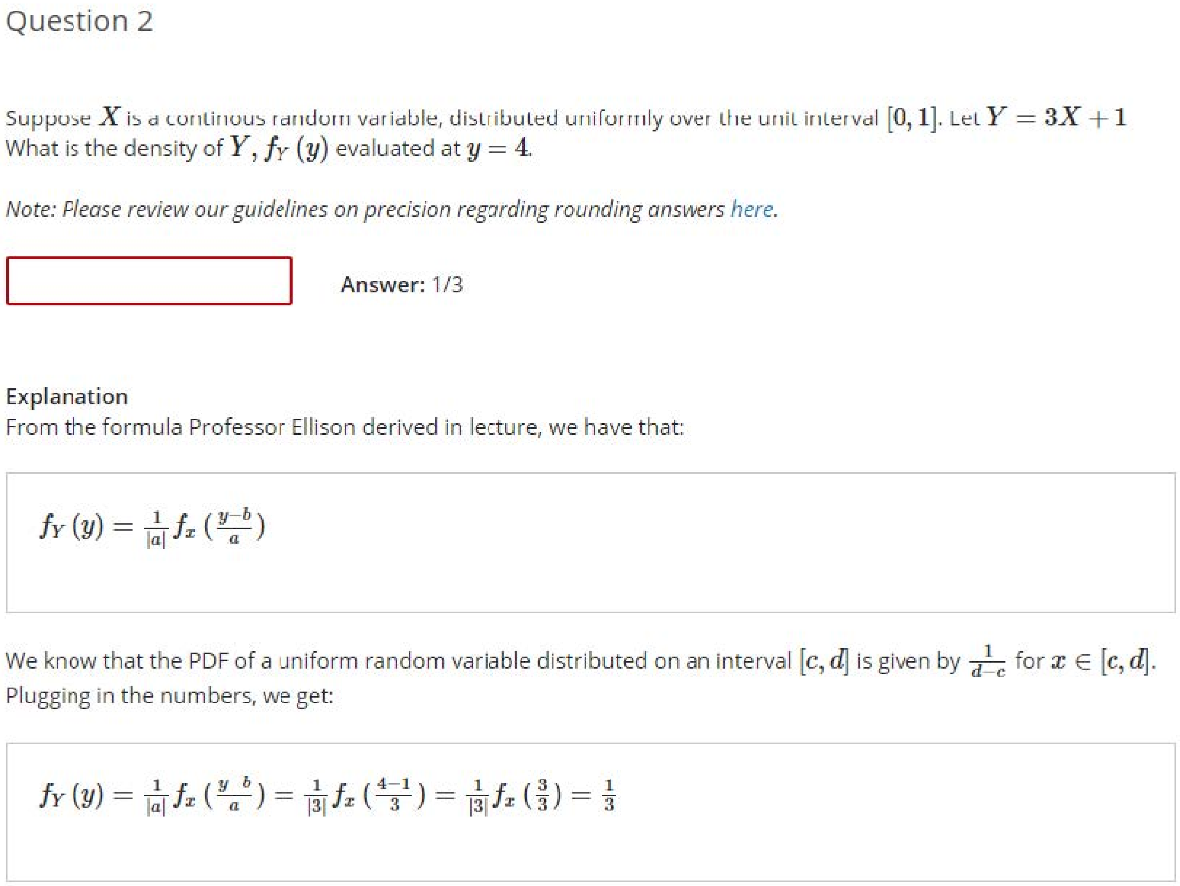


Question 2

Suppose X is a continous random variable, distributed uniformly over the unit interval $[0,1]$. Let $Y = 3X + 1$
What is the density of Y, $f_Y(y)$ evaluated at $y = 4$.

Note: Please review our guidelines on precision regarding rounding answers here.

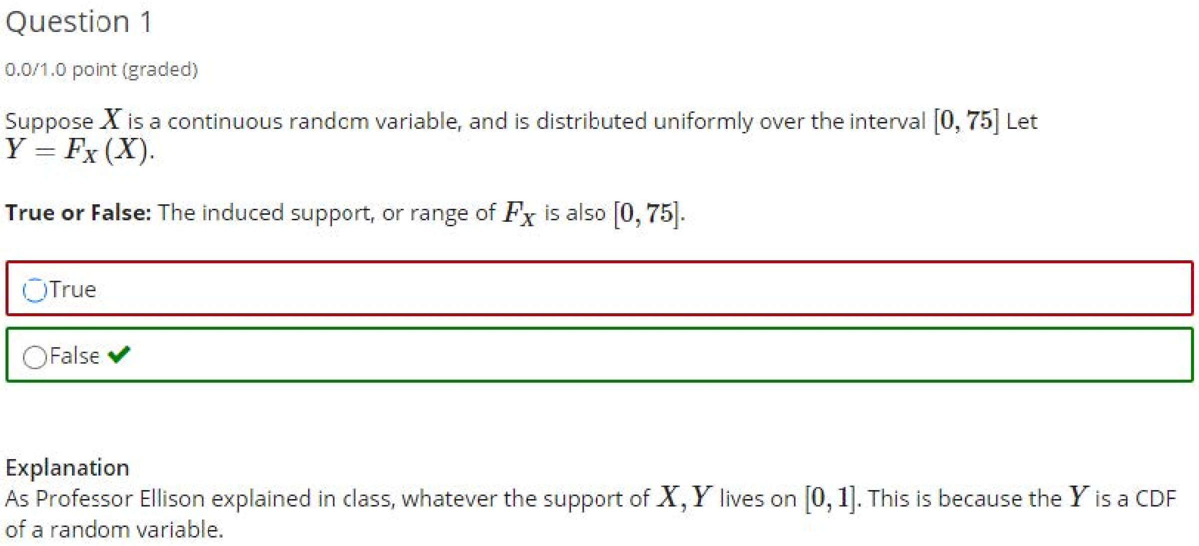
Answer: 1/3

Explanation
From the formula Professor Ellison derived in lecture, we have that:

$$f_Y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right)$$

We know that the PDF of a uniform random variable distributed on an interval $[c,d]$ is given by $\frac{1}{d-c}$ for $x \in [c,d]$.
Plugging in the numbers, we get:

$$f_Y(y) = \frac{1}{|a|} f_x\left(\frac{y\ \ b}{a}\right) = \frac{1}{|3|} f_x\left(\frac{4-1}{3}\right) = \frac{1}{|3|} f_x\left(\frac{3}{3}\right) = \frac{1}{3}$$



Question 1

0.0/1.0 point (graded)

Suppose X is a continuous random variable, and is distributed uniformly over the interval $[0,75]$ Let
$Y = F_X(X)$.

True or False: The induced support, or range of F_X is also $[0,75]$.

○ True

○ False ✓

Explanation
As Professor Ellison explained in class, whatever the support of X, Y lives on $[0,1]$. This is because the Y is a CDF of a random variable.

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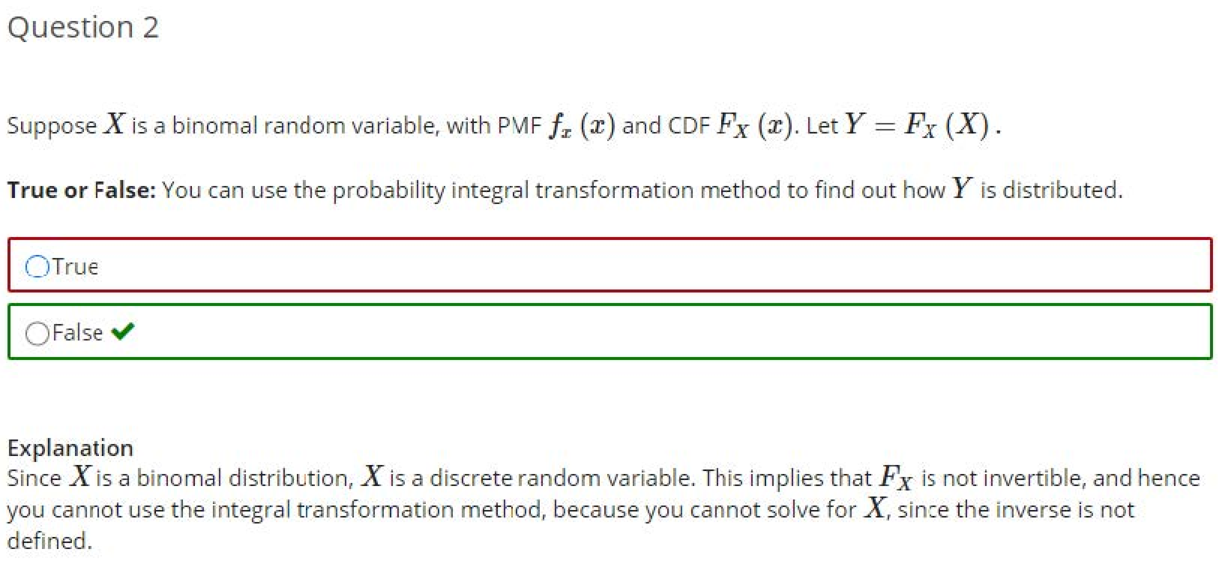


Question 2

Suppose X is a binomal random variable, with PMF $f_x(x)$ and CDF $F_X(x)$. Let $Y = F_X(X)$.

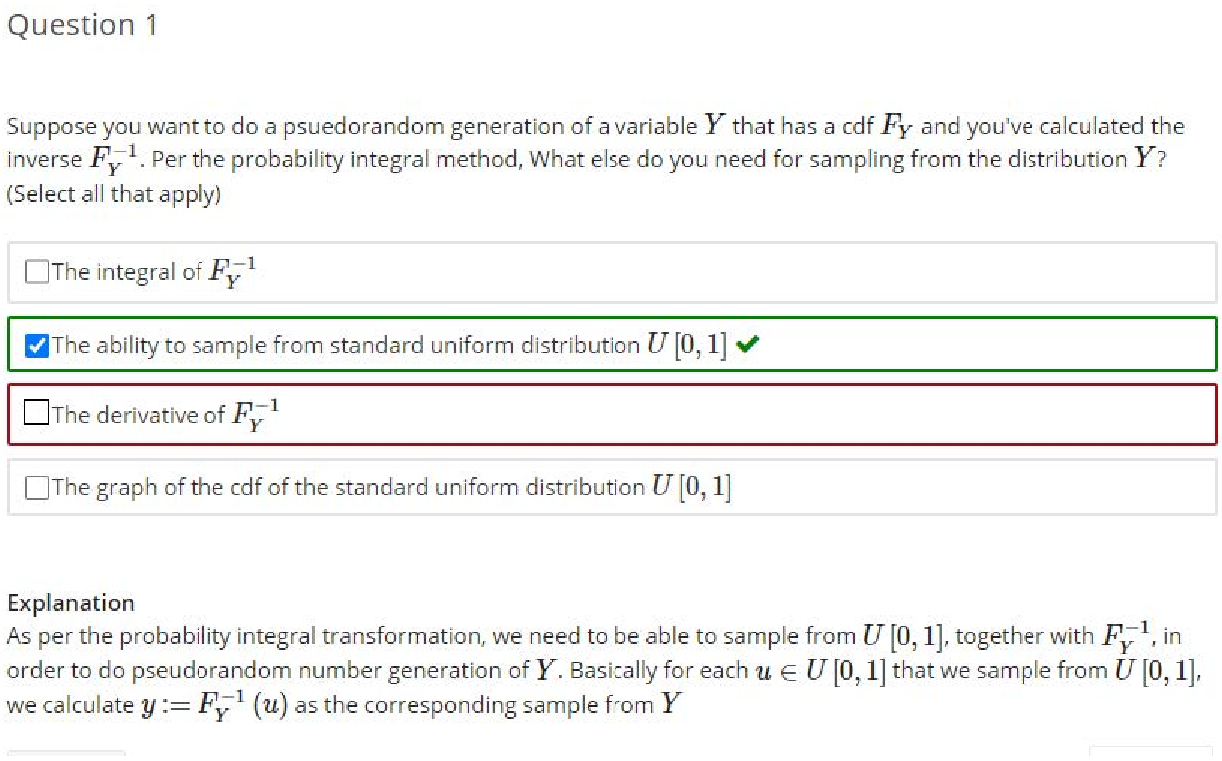
True or False: You can use the probability integral transformation method to find out how Y is distributed.

○ True

○ False ✓

Explanation

Since X is a binomal distribution, X is a discrete random variable. This implies that F_X is not invertible, and hence you cannot use the integral transformation method, because you cannot solve for X, since the inverse is not defined.



Question 1

Suppose you want to do a psuedorandom generation of a variable Y that has a cdf F_Y and you've calculated the inverse F_Y^{-1}. Per the probability integral method, What else do you need for sampling from the distribution Y? (Select all that apply)

☐ The integral of F_Y^{-1}

☑ The ability to sample from standard uniform distribution $U[0,1]$ ✓

☐ The derivative of F_Y^{-1}

☐ The graph of the cdf of the standard uniform distribution $U[0,1]$

Explanation

As per the probability integral transformation, we need to be able to sample from $U[0,1]$, together with F_Y^{-1}, in order to do pseudorandom number generation of Y. Basically for each $u \in U[0,1]$ that we sample from $U[0,1]$, we calculate $y := F_Y^{-1}(u)$ as the corresponding sample from Y

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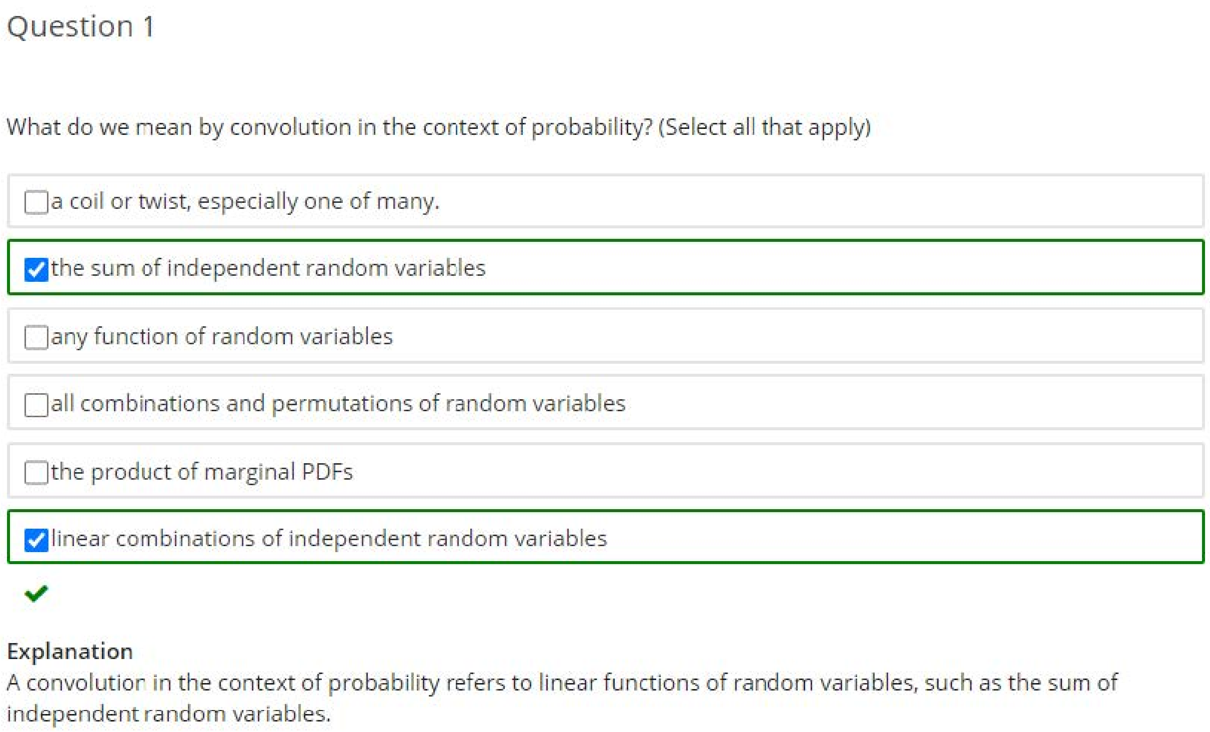


Question 1

What do we mean by convolution in the context of probability? (Select all that apply)

☐ a coil or twist, especially one of many.

☑ the sum of independent random variables

☐ any function of random variables

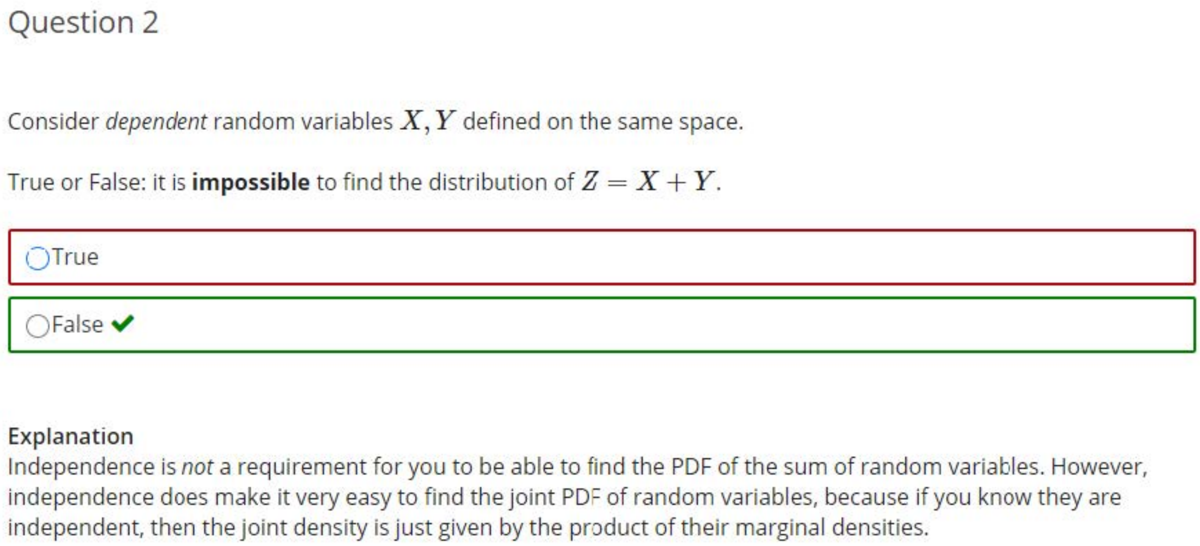
☐ all combinations and permutations of random variables

☐ the product of marginal PDFs

☑ linear combinations of independent random variables

✓

Explanation
A convolution in the context of probability refers to linear functions of random variables, such as the sum of independent random variables.



Question 2

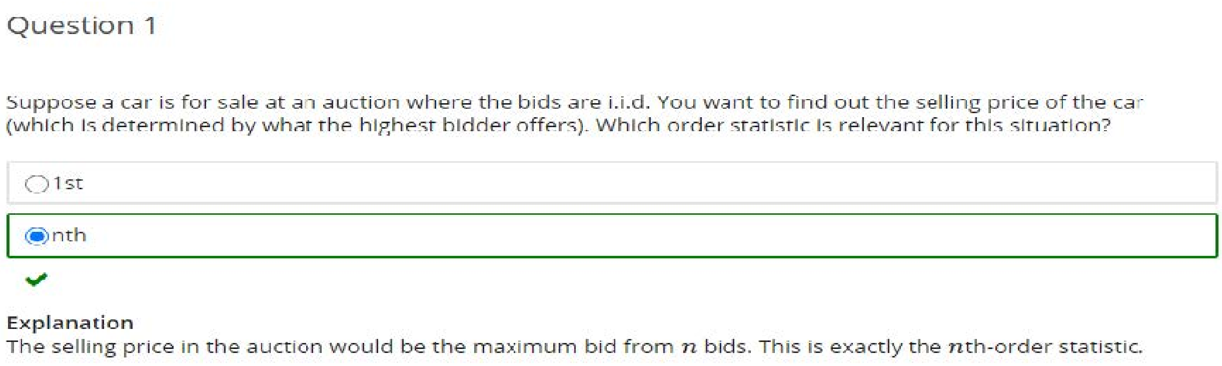
Consider *dependent* random variables X, Y defined on the same space.

True or False: it is **impossible** to find the distribution of $Z = X + Y$.

○ True

○ False ✓

Explanation
Independence is *not* a requirement for you to be able to find the PDF of the sum of random variables. However, independence does make it very easy to find the joint PDF of random variables, because if you know they are independent, then the joint density is just given by the product of their marginal densities.



Question 1

Suppose a car is for sale at an auction where the bids are i.i.d. You want to find out the selling price of the car (which is determined by what the highest bidder offers). Which order statistic is relevant for this situation?

○ 1st

● nth

✓

Explanation
The selling price in the auction would be the maximum bid from n bids. This is exactly the nth-order statistic.

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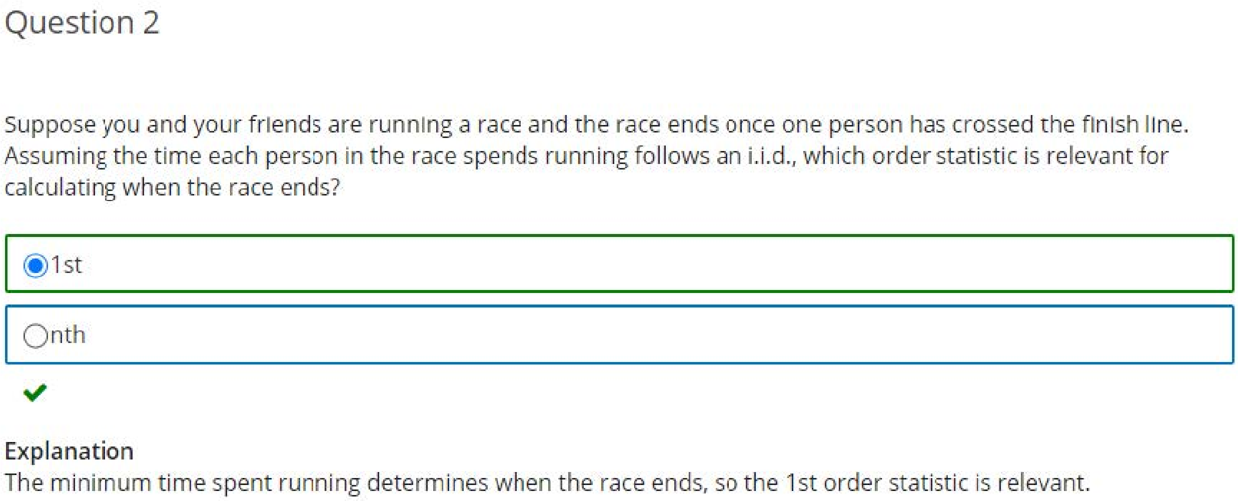


Question 2

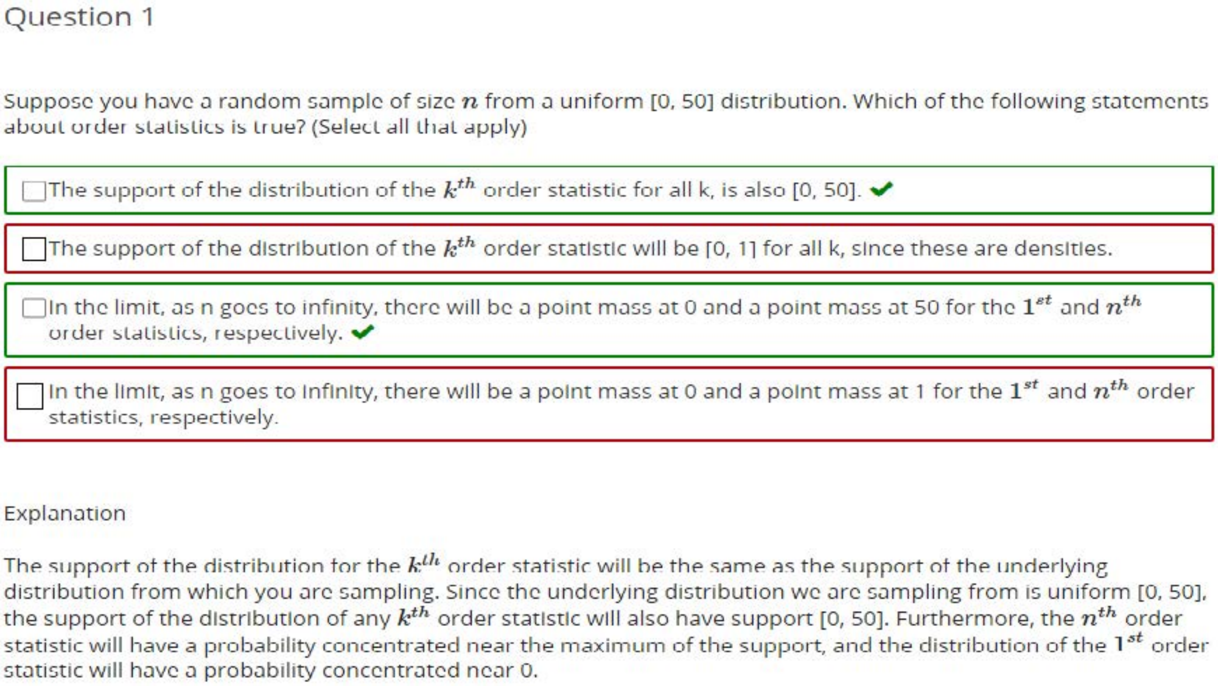
Suppose you and your friends are running a race and the race ends once one person has crossed the finish line. Assuming the time each person in the race spends running follows an i.i.d., which order statistic is relevant for calculating when the race ends?

● 1st

○ nth

✓

Explanation
The minimum time spent running determines when the race ends, so the 1st order statistic is relevant.



Question 1

Suppose you have a random sample of size n from a uniform [0, 50] distribution. Which of the following statements about order statistics is true? (Select all that apply)

☐ The support of the distribution of the k^{th} order statistic for all k, is also [0, 50]. ✓

☐ The support of the distribution of the k^{th} order statistic will be [0, 1] for all k, since these are densities.

☐ In the limit, as n goes to infinity, there will be a point mass at 0 and a point mass at 50 for the 1^{st} and n^{th} order statistics, respectively. ✓

☐ In the limit, as n goes to infinity, there will be a point mass at 0 and a point mass at 1 for the 1^{st} and n^{th} order statistics, respectively.

Explanation

The support of the distribution for the k^{th} order statistic will be the same as the support of the underlying distribution from which you are sampling. Since the underlying distribution we are sampling from is uniform [0, 50], the support of the distribution of any k^{th} order statistic will also have support [0, 50]. Furthermore, the n^{th} order statistic will have a probability concentrated near the maximum of the support, and the distribution of the 1^{st} order statistic will have a probability concentrated near 0.

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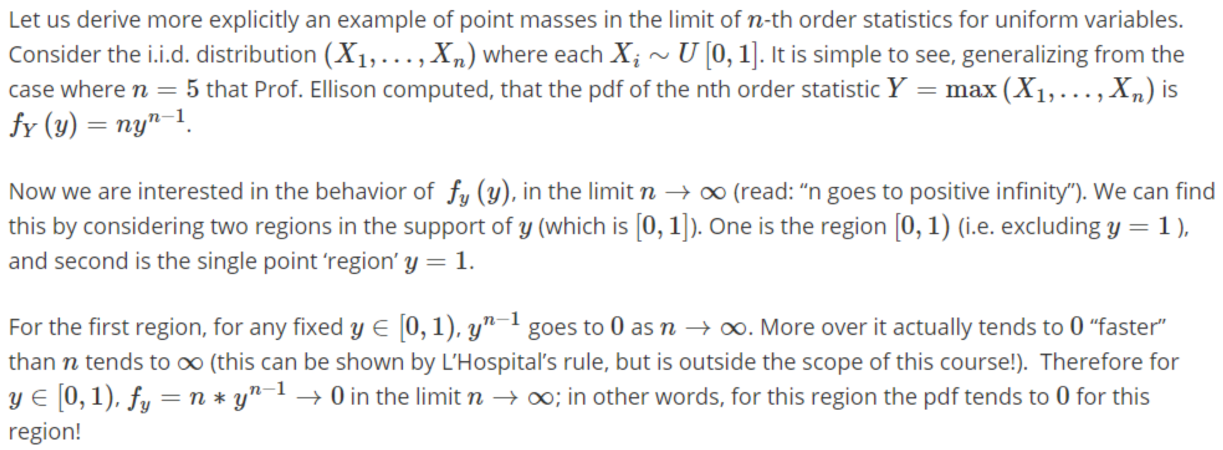
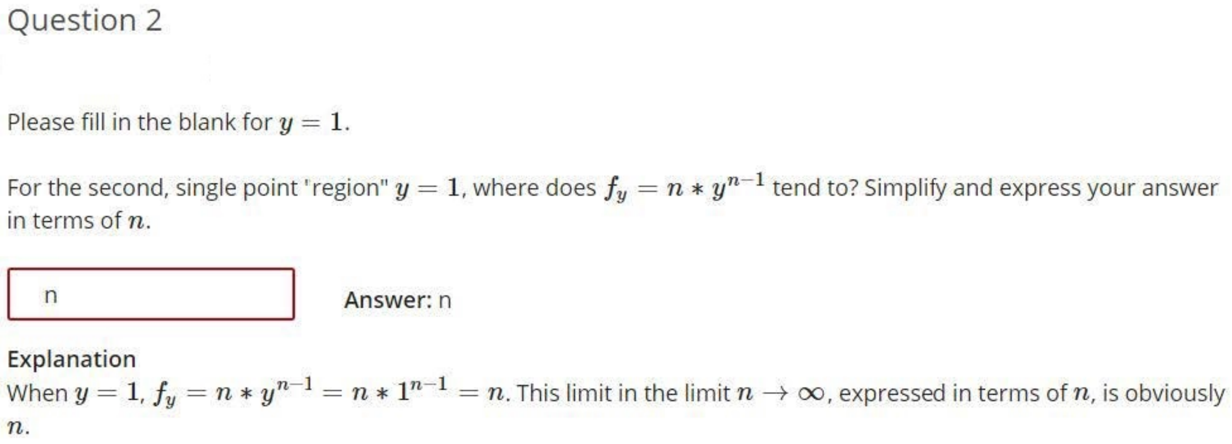


Let us derive more explicitly an example of point masses in the limit of n-th order statistics for uniform variables. Consider the i.i.d. distribution $(X_1, \ldots, X_n)$ where each $X_i \sim U\,[0,1]$. It is simple to see, generalizing from the case where $n = 5$ that Prof. Ellison computed, that the pdf of the nth order statistic $Y = \max\,(X_1, \ldots, X_n)$ is $f_Y\,(y) = ny^{n-1}$.

Now we are interested in the behavior of $f_y\,(y)$, in the limit $n \to \infty$ (read: "n goes to positive infinity"). We can find this by considering two regions in the support of y (which is $[0,1]$). One is the region $[0,1)$ (i.e. excluding $y = 1$), and second is the single point 'region' $y = 1$.

For the first region, for any fixed $y \in [0,1)$, y^{n-1} goes to 0 as $n \to \infty$. More over it actually tends to 0 "faster" than n tends to ∞ (this can be shown by L'Hospital's rule, but is outside the scope of this course!). Therefore for $y \in [0,1)$, $f_y = n * y^{n-1} \to 0$ in the limit $n \to \infty$; in other words, for this region the pdf tends to 0 for this region!



Question 2

Please fill in the blank for $y = 1$.

For the second, single point 'region' $y = 1$, where does $f_y = n * y^{n-1}$ tend to? Simplify and express your answer in terms of n.

n

Answer: n

Explanation

When $y = 1$, $f_y = n * y^{n-1} = n * 1^{n-1} = n$. This limit in the limit $n \to \infty$, expressed in terms of n, is obviously n.